Duality in fuzzy number linear programming by use of a certain linear ranking function

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Abstract

We explore some duality properties in fuzzy number linear programming problems. By use of a linear ranking function we introduce the dual of fuzzy number linear programming primal problems. We then present several duality results.

1. Introduction

The concept of decision making in fuzzy environment was first proposed by Bellman and Zadeh [2]. Subsequently, Tanaka et al. [18] made use of this concept in mathematical programming (see also [19]). Fuzzy linear programming problem with fuzzy coefficients was proposed by Negoita [14]. A formulation of fuzzy linear programming with fuzzy constraints and a solution method were given by Tanaka and Asai [16]. Maleki et al. [12] introduced a linear programming problem with fuzzy variables and proposed a method for solving it. Fang and Hu [5] consider linear programming with fuzzy constraint coefficients (see also [7]). Gasimov and Yenilmez [9] discuss solution of fuzzy linear programming problems using linear membership functions (see also [8]). Maleki [11] used a certain ranking function to solve fuzzy linear programming problems. He also introduced a new method for solving linear programming problems with vagueness in constraints using a linear ranking function. Mishmast et al. [13] introduced the lexicographic ranking function to order fuzzy numbers and solved fuzzy number linear programming problems by lexicographic ranking function. Here, we first review the fundamental notions of fuzzy sets and fuzzy numbers. Then we consider a linear ranking function, similar to the ranking function proposed by Roubens [6], to order fuzzy numbers. Then, we introduce the dual of fuzzy number linear programming problems. Finally, we investigate and characterize several properties for the fuzzy number linear programming problem and its dual.
2. Definitions and notations

We review the fundamental notions of fuzzy set theory, initiated by Bellman and Zadeh [2], to be used throughout this note (see also [4] or [20]). Below, we give definitions and notations taken from Bezdek [3].

**Definition 2.1.** Let $X$ be the universal set. $\widetilde{A}$ is called a fuzzy set in $X$ if $\widetilde{A}$ is a set of ordered pairs 
$$ \widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) | x \in X\}, $$ 
where $\mu_{\widetilde{A}}(x)$ is the membership function of $x$ in $\widetilde{A}$.

**Remark 2.1.** The membership function of $\widetilde{A}$ specifies the degree of membership of element $x$ in fuzzy set $\widetilde{A}$ (in fact, $\mu_{\widetilde{A}}$ shows the degree that $x$ belongs to $\widetilde{A}$).

**Definition 2.2.** The $\alpha$-level set of $\widetilde{A}$ is the set 
$$ \widetilde{A}_\alpha = \{x \in \mathbb{R} | \mu_{\widetilde{A}}(x) \geq \alpha\}, $$ 
where $\alpha \in (0, 1]$. The lower and upper bounds of any $\alpha$-level set $\widetilde{A}_\alpha$ are represented by finite numbers $\inf_{x \in \widetilde{A}_\alpha}$ and $\sup_{x \in \widetilde{A}_\alpha}$.

**Definition 2.3.** The support of a fuzzy set $\widetilde{A}$ is a set of elements in $X$ for which $\mu_{\widetilde{A}}(x)$ is positive, that is, 
$$ \text{supp} \widetilde{A} = \{x \in X | \mu_{\widetilde{A}}(x) > 0\}. $$

**Definition 2.4.** A fuzzy set $\widetilde{A}$ is convex if 
$$ \mu_{\widetilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{A}}(y)\}, \quad \forall x, y \in X \text{ and } \lambda \in [0, 1]. $$

**Definition 2.5.** A convex fuzzy set $\widetilde{A}$ on $\mathbb{R}$ is a fuzzy number if the following conditions hold:

(a) Its membership function is piecewise continuous.

(b) There exist three intervals $[a, b]$, $[b, c]$ and $[c, d]$ such that $\mu_{\widetilde{A}}$ is increasing on $[a, b]$, equal to 1 on $[b, c]$, decreasing on $[c, d]$ and equal to 0 elsewhere.

**Remark 2.2.** In the above definition, we say interval $[b, c]$ is the modal set of fuzzy number $\widetilde{A}$.

**Definition 2.6.** Let $\widetilde{A} = (a_L, a_U, \alpha, \beta)$ denote the trapezoidal fuzzy number, where $[a_L - \alpha, a_U + \beta]$ is the support of $\widetilde{A}$ and $[a_L, a_U]$ its modal set. See Fig. 1.

**Remark 2.3.** We denote the set of all trapezoidal fuzzy numbers by $\mathcal{F}(\mathbb{R})$. If $a = a_L = a_U$ then we obtain a triangular fuzzy number, and we show it with $\widetilde{A} = (a, \alpha, \beta)$. See Fig. 2.

We next define arithmetic on fuzzy numbers. Let $\tilde{a} = (a_L, a_U, \alpha, \beta)$ and $\tilde{b} = (b_L, b_U, \gamma, \delta)$ be two trapezoidal fuzzy numbers. Define

![Fig. 1. Trapezoidal fuzzy number.](image-url)
Several methods for solving fuzzy linear programming problems can be seen in Fang [5], Lai and Hwang [10], Maleki et al. [12], Shoacheng [15], and Tanaka and Ichihashi [17]. One of the most convenient of these methods is based on the concept of comparison of fuzzy numbers by use of ranking functions [8,11]. In fact, an efficient approach for ordering the elements of $F(R)$ is to define a ranking function $\mathcal{R} : F(R) \rightarrow \mathbb{R}$ which maps each fuzzy number into the real line, where a natural order exists.

We define orders on $F(R)$ by
\[
\bar{a} P \bar{b} \quad \text{iff} \quad \mathcal{R}(\bar{a}) > \mathcal{R}(\bar{b}),
\]
\[
\bar{a} \not P \bar{b} \quad \text{iff} \quad \mathcal{R}(\bar{a}) > \mathcal{R}(\bar{b}),
\]
\[
\bar{a} = \bar{b} \quad \text{iff} \quad \mathcal{R}(\bar{a}) = \mathcal{R}(\bar{b}),
\]
where $\bar{a}$ and $\bar{b}$ are in $F(R)$. Also weh write $\bar{a} \leq \bar{b}$ if and only if $\bar{b} \geq \bar{a}$.

The following lemma is now immediate.

**Lemma 3.1.** Let $\mathcal{R}$ be any linear ranking function. Then
\[(i) \quad \bar{a} P \bar{b} \quad \text{iff} \quad \bar{a} - \bar{b} \geq 0 \quad \text{iff} \quad -\bar{b} \geq -\bar{a}.
\]
\[(ii) \quad \text{If} \quad \bar{a} P \bar{b} \text{ and } \bar{c} P \bar{d}, \text{ then } \bar{a} + \bar{c} P \bar{b} + \bar{d}.
\]

We restrict our attention to linear ranking functions, that is, a ranking function $\mathcal{R}$ such that
\[
\mathcal{R}(k\bar{a} + \bar{b}) = k\mathcal{R}(\bar{a}) + \mathcal{R}(\bar{b})
\]
for any $\bar{a}$ and $\bar{b}$ belonging to $F(R)$ and any $k \in \mathbb{R}$.

Here, we introduce a linear ranking function that is similar to the ranking function adopted by Maleki [11]. For a trapezoidal fuzzy number $\bar{a} = (a^L, a^U, x, \beta)$, we use ranking function as follows:
\[
\mathcal{R}(\bar{a}) = \int_0^1 (\inf \bar{a}_x + \sup \bar{a}_x) \, dx.
\]

This reduces to
\[
\mathcal{R}(\bar{a}) = a^L + a^U + \frac{1}{2}(\beta - x).
\]
Then, for trapezoidal fuzzy numbers $\tilde{a}$ and $\tilde{b}$, we have

$$\tilde{a} \succcurlyeq \tilde{b} \text{ if and only if } a^L + a^U + \frac{1}{2}(\beta - \alpha) \succcurlyeq b^L + b^U + \frac{1}{2}(\theta - \gamma).$$

(3.7)

4. Fuzzy number linear programming problems

Authors who use ranking functions for comparison of fuzzy numbers usually define a crisp model which is equivalent to the fuzzy number linear programming problem and then use optimal solution of this model as the optimal solution of fuzzy number linear programming problem. We now define fuzzy number linear programming problems and the corresponding crisp models.

4.1. Formulation of the fuzzy number linear programming problem

Definition 4.1. A fuzzy number linear programming problem (FNLPP) is defined as follows:

$$\max_{\tilde{z} = \tilde{c}x} \quad \text{ s.t. } \quad \tilde{A}x \leq \tilde{b}, \quad x \geq 0,$$

(4.1)

where ‘$=$’ and ‘$\leq$’ mean equality and inequality with respect to the ranking function $R, \tilde{A} = (\tilde{a}_{ij})_{m \times n}$, $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_n)$, $\tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_m)^T$ and $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_j \in \mathcal{F}(\mathbb{R})$ for $i = 1, \ldots, m, j = 1, 2, \ldots, n$.

Definition 4.2. Any $x$ which satisfies the set of constraints of FNLPP is called a feasible solution. Let $Q$ be the set of all feasible solutions of FNLPP. We say that $x^0 \in Q$ is an optimal feasible solution for FNLPP if $\tilde{c}x \leq \tilde{c}x^0$ for all $x \in Q$.

Definition 4.3. We say that the real number $a$ corresponds to the fuzzy number $\tilde{a}$, with respect to a given linear ranking function $R$, if $a = R(\tilde{a})$.

The following theorem shows that any FNLPP can be reduced to a linear programming problem (see Maleki [11] and Maleki et al. [12]).

Theorem 4.1. The following linear programming problem (LPP) and the FNLPP in (4.1) are equivalent:

$$\max \quad z = cx \quad \text{ s.t. } \quad \begin{cases} Ax \leq b, \\ x \geq 0, \end{cases}$$

or

$$\max \quad z = \sum_{j=1}^{n} c_j x_j \quad \text{ s.t. } \quad \begin{cases} \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m, \\ x_j \geq 0 \quad j = 1, \ldots, n, \end{cases}$$

(4.2)

where $a_{ij}, b_i, c_j$ are real numbers corresponding to the fuzzy numbers $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_j$ with respect to a given linear ranking function $R$, respectively.

Remark 4.1. The above theorem shows that the sets of all feasible solutions of FNLPP and LPP are the same. Also if $\tilde{x}$ is an optimal feasible solution for FNLPP, then $\tilde{x}$ is an optimal feasible solution for LPP.

Corollary 4.1. If LPP does not have a solution then FNLPP does not have a solution either.

For an illustration of Theorem 4.1 consider an example used by Maleki [11].
Example 4.1. Consider the following FNLPP:

\[
\begin{align*}
\text{max} \quad z &= (2, 3, 1, 1)x_1 + (3, 4, 1, 2)x_2 \\
\text{s.t.} \quad (1, 2, 1, 1)x_1 + (2, 3, 1, 2)x_2 &\leq (5, 6, 2, 2), \\
&\quad (2, 3, 1, 3)x_1 + (1, 2, 1, 1)x_2 \leq (4, 6, 2, 1), \\
&\quad x_1, x_2 \geq 0,
\end{align*}
\]

where \((a^L, a^U, x, \beta)\) is a trapezoidal fuzzy number.

We apply the ranking function (3.6) to solve the above FNLPP. The problem reduces to

\[
\begin{align*}
\text{max} \quad z &= 5x_1 + 7.5x_2 \\
\text{s.t.} \quad 3x_1 + 5.5x_2 &\leq 11, \\
&\quad 6x_1 + 3x_2 \leq 9.5, \\
&\quad x_1, x_2 \geq 0.
\end{align*}
\]

We write the above linear programming problem in the standard form

\[
\begin{align*}
\text{max} \quad z &= 5x_1 + 7.5x_2 \\
\text{s.t.} \quad 3x_1 + 5.5x_2 + x_3 &= 11, \\
&\quad 6x_1 + 3x_2 + x_4 = 9.5, \\
&\quad x_1, x_2, x_3, x_4 \geq 0,
\end{align*}
\]

where new variables \(x_3\) and \(x_4\) are called slack variables.

The optimal solution, rounded to three decimal places, is \(x_1^* = 0.802\), \(x_2^* = 1.562\) and \(z^* = (6.292, 2.365, 8.656, 3.927)\).

4.2. Basic feasible solution

Consider the FNLPP problem

\[
\begin{align*}
\text{max} \quad \tilde{z} &= \tilde{c}x \\
\text{s.t.} \quad \tilde{A}x &= \tilde{b}, \\
&\quad x \geq 0,
\end{align*}
\]

where the parameters of the problem are as defined in (4.1).

Let \(A = [a_{ij}]_{m \times n} = [\mathscr{R}(\tilde{a}_{ij})] = \mathscr{R}(\tilde{A})\). Assume \(\text{rank}(A) = m\). Partition \(A\) as \([BN]\) where \(B, m \times m\), is nonsingular. It is obvious that \(\text{rank}(B) = m\) and \(\tilde{B} = \mathscr{R}(\tilde{B})\), where \(\tilde{B}\) is the fuzzy matrix in \(\tilde{A}\) corresponding to \(B\). Let \(y_j\) be the solution to \(By = a_j\). It is apparent that the basic solution

\[
x_B = (x_{B_1}, \ldots, x_{B_n})^T = B^{-1}b, \quad x_N = 0
\]

is a solution of \(Ax = b\), where \(b = \mathscr{R}(\tilde{b})\). If \(x_B \geq 0\), then the basic solution is feasible and the corresponding fuzzy objective value is: \(\tilde{z} = \tilde{c}_B x_B\), where \(\tilde{c}_B = (\tilde{c}_{B_1}, \ldots, \tilde{c}_{B_n})\). Now, corresponding to every nonbasic variable \(x_j\), \(1 \leq j \leq n, j \neq B_i, i = 1, \ldots, m\), define

\[
\tilde{z}_j = \tilde{c}_B y_j = \tilde{c}_B B^{-1}a_j.
\]

The following theorem characterizes optimal solutions. The result corresponds to the so-called nondegenerate problems, where all basic variables corresponding to every basis \(B\) are nonzero (and hence positive).
Theorem 4.2. Assume the FNLPP is nondegenerate. A basic feasible solution \( x_B = B^{-1}b, x_N = 0 \) is optimal to (4.5) if and only if \( \tilde{z}_j \geq \tilde{c}_j \) for all \( j, 1 \leq j \leq n \).

Proof. The “if ” part has been stated and proved in [12]. Here, we prove the “only if ” part. Suppose that \( x = (x_B^T, x_N^T)^T \) is an optimal feasible solution to (4.5), where \( x_B = B^{-1}b, x_N = 0 \), corresponding to an optimal basis \( B \). Then, the corresponding optimal objective value is

\[
\tilde{z} = \tilde{c}x = \tilde{c}_Bx_B + \tilde{c}_N x_N = \tilde{c}_B B^{-1}b.
\]

Now, since \( x \) is feasible, we have \( x \geq 0 \), and

\[
b = Ax = Bx_B + Nx_N.
\]

Hence, we can rewrite (4.7) as follows:

\[
x_B = B^{-1}b - B^{-1}Nx_N.
\]

Substituting (4.8) in (4.6), we obtain

\[
\tilde{z} = \tilde{c}x = \tilde{c}_Bx_B + \tilde{c}_N x_N = \tilde{c}_B B^{-1}b - (\tilde{c}_B B^{-1} N - \tilde{c}_N)x_N = \tilde{c}_B B^{-1}b - \sum_{j \notin B_i} (\tilde{c}_B B^{-1} a_j - \tilde{c}_j)x_j.
\]

Thus

\[
\tilde{z} = \tilde{z}_s - \sum_{j \notin B_i} (\tilde{z}_j - \tilde{c}_j)x_j.
\]

Now, from (4.9) it is obvious that if for any nonbasic variable \( x_j \) we have \( \tilde{z}_j < \tilde{c}_j \), then we can enter \( x_j \) into the basis and obtain \( \tilde{z} < \tilde{z} \) (because the problem is nondegenerate and \( x_j > 0 \) in the new basis). This is contradictory to \( \tilde{z} \) being optimal and hence we must have \( \tilde{z}_j \geq \tilde{c}_j \). \( \square \)

5. Duality in fuzzy number linear programming

Similar to the duality theory in linear programming (see for example, Bazaraa et al. [1]), for every FNLPP, there is an associated problem which satisfies some important properties. We shall call this related FNLPP the DFNLPP.

5.1. Formulation of the dual problem

For the FNLPP

\[
\begin{align*}
\text{max} & \quad \tilde{c}x \\
\text{s.t.} & \quad \tilde{A}x \leq \tilde{b}, \\
& \quad x \geq 0
\end{align*}
\]

define the dual fuzzy number linear programming problem (DFNLPP) as

\[
\begin{align*}
\text{min} & \quad w\tilde{b} \\
\text{s.t.} & \quad \tilde{w}A \geq \tilde{c}, \\
& \quad w \geq 0.
\end{align*}
\]

Note that there is exactly one dual variable (of the form \( \geq 0 \)) for each FNLPP constraint of the form \( \leq \) and exactly one dual constraint (of the form \( \geq \)) for each variable of the form \( \geq \) in FNLPP.
Example 5.1. Consider the given FNLPP in Example 4.1. The dual to this problem follows:

\[
\begin{align*}
\text{min} \quad & \tilde{u} = (5, 6, 2, 2)w_1 + (4, 6, 2, 1)w_2 \\
\text{s.t.} \quad & (1, 2, 1, 1)w_1 + (2, 3, 1, 3)w_2 \geq (2, 3, 1, 1), \\
& (2, 3, 1, 2)w_1 + (1, 2, 1, 1)w_2 \geq (3, 4, 1, 2), \\
& w_1, w_2 \geq 0.
\end{align*}
\]

Now if we apply the ranking function (3.6), we have

\[
\begin{align*}
\text{min} \quad & u = 11w_1 + 9.5w_2 \\
\text{s.t.} \quad & 3w_1 + 6w_2 \geq 5, \\
& 5.5w_1 + 3w_2 \geq 7.5, \\
& w_1, w_2 \geq 0.
\end{align*}
\]

The optimal solution, rounded to three decimal places, is \( w_1^* = 1.250, \ w_2^* = 0.208 \) and \( \tilde{u}_* = (7.083, 8.750, 2.917, 2.708) \). Applying the ranking function (3.6), we then obtain \( \mathcal{R}(\tilde{u}_*) = 15.729 \). Likewise, from Example 4.1, we have \( \mathcal{R}(\tilde{z}_*) = 15.729 \).

Remark 5.1. It is important to note that the FNLPP may have alternative optimal solutions, but the value of the ranking function for the fuzzy value of the objective function corresponding to all optimal solutions is unique.

5.2. The relationships between FNLPP and DFNLPP

We shall discuss here the relationships between the fuzzy number linear programming problem and its corresponding dual.

Lemma 5.1. Dual of DFNLPP is FNLPP.

Proof. Use Lemma 3.1 and the definition of DFNLPP. \( \square \)

Remark 5.2. Lemma 5.1 indicates that the duality results can be applied to any of the primal or dual problems posed as the primal problem.

Theorem 5.1 (The weak duality property). If \( x_0 \) and \( w_0 \) are feasible solutions to FNLPP and DFNLPP, respectively, then \( \tilde{c}x_0 \leq \tilde{w}_0 \tilde{b} \).

Proof. Multiplying \( \tilde{A}x_0 \leq \tilde{b} \) on the left by \( w_0 \geq 0 \) and \( w_0\tilde{A} \geq \tilde{c} \) on the right by \( x_0 \geq 0 \), we get \( \tilde{c}x_0 \leq \tilde{w}_0\tilde{A}x_0 \leq \tilde{w}_0\tilde{b} \). \( \square \)

Remark 5.3. The value of the ranking function for the fuzzy value of the objective function at any feasible solution to FNLP is always lower than or equal to the value of the ranking function for the fuzzy value of the objective function for any feasible solution to DFNLPP.

As an illustration of the application of this theorem, suppose that in Examples 4.1 and 5.1 we select the feasible FNLPP and DFNLPP solutions \( x_0 = (1, 1)^T \) and \( w_0 = (2, 1) \). Then \( \tilde{c}x_0 = (5, 7, 2, 3) \) and \( \tilde{w}_0\tilde{b} = (14, 18, 6, 5) \). Now if we apply the ranking function (3.6), we obtain \( \mathcal{R}(\tilde{c}x_0) = 12.5 < \mathcal{R}(\tilde{w}_0\tilde{b}) = 31.5 \). Thus at the optimal solution, the value of the ranking function for the fuzzy value of the objective function lies between 12.5 and 31.5.
The following corollaries are immediate consequences of Theorem 5.1

**Corollary 5.1.** If \( x_0 \) and \( w_0 \) are feasible solutions to FNLPP and DFNLPP, respectively, and \( \tilde{c}x_0 = w_0 \tilde{b} \), then \( x_0 \) and \( w_0 \) are optimal solutions to their respective problems.

The following corollary relates unboundeness of one problem to infeasibility of the other. We use the definition below.

**Definition 5.1.** We say FNLPP (or DFNLPP) is unbounded if feasible solutions exist with arbitrary large (or small) ranking values for the fuzzy objective function.

**Corollary 5.2.** If either problems is unbounded, then the other problem has no feasible solution.

We now state and prove the main duality result.

**Theorem 5.2 (Strong duality).** If any one of the FNLPP or DFNLPP has an optimal solution, then both problems have optimal solutions and the two optimal values of ranking functions for the fuzzy objectives are equal.

**Proof.** Because of Lemma 5.1, it suffices to show the result assuming the existence of optimal solution for the FNLPP. Assume that the FNLPP has an optimal solution, and \( \text{rank}(A) = m \). Let \( y \geq 0 \) be the slack variables for the constraints \( Ax \leq \tilde{b} \). The new equivalent problem to the FNLPP is

\[
\begin{align*}
\max \quad & \tilde{z} = \tilde{c}x + 0y \\
\text{s.t.} \quad & \tilde{A}x + y = \tilde{b}, \\
& x \geq 0, \\
& y \geq 0.
\end{align*}
\]

(5.3)

Assume \( B \) is the basis matrix and \( x_* = (x^T_B \ 0)^T \) is the basic optimal solution corresponding to the FNLPP. From Theorem 4.2 we have

\[
c_B B^{-1}a_j - c_j \geq 0, \quad j = 1, \ldots, n, n + 1, \ldots, n + m,
\]

or equivalently

\[
c_B B^{-1}a_j \geq c_j, \quad j = 1, \ldots, n,
\]

\[
c_B B^{-1}e_i \geq 0, \quad i = 1, \ldots, m.
\]

Hence we must have

\[
c_B B^{-1}A \geq c,
\]

\[
c_B B^{-1} \geq 0.
\]

Thus

\[
c_B B^{-1}R(\tilde{A}) \geq R(\tilde{c}),
\]

\[
c_B B^{-1} \geq 0.
\]

Let \( w_* = c_B B^{-1} \). Using the above inequalities, we can write

\[
w_* \tilde{A} \geq \tilde{c},
\]

\[
w_* \geq 0.
\]

Thus, \( w_* \) is a feasible solution to the DFNLPP and

\[
w_* R(\tilde{b}) = w_* b = c_B B^{-1}b = c_B x_B = c x_* = R(\tilde{c}) x_*
\]
and hence
\[ w, \bar{b} = \bar{c}x. \]
Therefore, the result follows immediately from Corollary 5.1.

For an illustration of the above theorem consider the FNLPP and DFNLP given in Examples 4.1 and 5.1, respectively. We see that the basic optimal solution for the FNLPP is \( x_B = (x_1, x_2)^T = (0.802, 1.562)^T \), with the basis matrix \( B = \begin{bmatrix} 3 & 11 & 2 \\ 4 & 6 & 3 \end{bmatrix} \). Hence, if we let \( w = c_B B^{-1} \), we obtain \( w = (1.250, 0.208) \), which is equal to the optimal solution of the DFNLP.

Using the results of the lemmas, corollaries, and remarks above we obtain the following important duality result.

**Theorem 5.3** (Fundamental theorem of duality). For any FNLPP and its corresponding DFNLP, exactly one of the following statements is true.

1. Both have optimal solutions \( x_* \) and \( w_* \) with \( \sim_c x = R \sim_w b \).
2. One problem is unbounded and the other is infeasible.
3. Both problems are infeasible.

Using the above results, (1) and (2) are obviously correct. We show (3) by an example.

**Example 5.2.** Consider FNLPP and its corresponding DFNLP as follows:

\[
\begin{align*}
\max & \quad \bar{z} = (1, 2, 5, 1)x_1 + (-1, 0, 2, 6)x_2 \\
\text{s.t.} & \quad (1, 2, 5, 1)x_1 + (-1, 1, 3, 1)x_2 \leq (-1, 1, 4, 2), \\
& \quad (0, 1, 6, 2)x_1 - (-1, 1, 3, 1)x_2 \leq (-2, -1, 3, 7), \\
& \quad x_1, x_2 \geq 0,
\end{align*}
\]

\[
\begin{align*}
\min & \quad \bar{u} = (-1, 1, 4, 2)w_1 + (-2, -1, 3, 7)w_2 \\
\text{s.t.} & \quad (1, 2, 5, 1)w_1 + (0, 1, 6, 2)w_2 \geq (1, 2, 5, 1), \\
& \quad (-1, 1, 3, 1)w_1 - (-1, 1, 3, 1)w_2 \geq (-1, 0, 2, 6), \\
& \quad w_1, w_2 \geq 0.
\end{align*}
\]

We see that both problems are infeasible.

We now state and prove an important result of duality theory, generally named as complementary slackness.

**Theorem 5.4** (Complementary slackness). Let \( x_* \) and \( w_* \) be any feasible solutions to FNLPP and its corresponding DFNLP. Then \( x_* \) and \( w_* \) are respectively optimal if and only if

\[
(w - \bar{c})x_* + w_*(\bar{b} - \bar{A}x_*) = 0. \tag{5.4}
\]

**Proof.** Suppose that \( x_* \) and \( w_* \) are feasible solutions to FNLPP and DFNLP, respectively. Therefore

\[
\bar{A}x_* \leq \bar{b}, \tag{5.5}
\]

or

\[
\bar{A}(\bar{A})x_* \leq \bar{A}(\bar{b}), \tag{5.6}
\]
where, with respect to the ranking function \( R \), we have
\[
R(A) = (R(\tilde{a}_{ij}))_{m \times n} \quad R(\tilde{b}) = (R(\tilde{b}_i))_{m \times 1},
\]
and
\[
\tilde{a}_{ij}, \quad \tilde{b}_i \in \mathcal{F}(\mathbb{R}), \quad \text{for } i = 1, \ldots, m, \quad j = 1, \ldots, n.
\]

By adding the nonnegative slack variables vector, we obtain
\[
R(A)x + u = R(\tilde{b}). \tag{5.7}
\]

Since \( w_* \) is a feasible solution to DFNLP, then
\[
w_*A \geq \tilde{\mathcal{c}}, \tag{5.8}
\]
or
\[
w_*R(\tilde{A}) \geq R(\tilde{c}). \tag{5.9}
\]
This constraint can be put in equality form by subtracting the nonnegative surplus variables vector leading to
\[
w_*R(\tilde{A}) - v = R(\tilde{c}). \tag{5.10}
\]

Multiplying (5.7) on the left by \( w_* \geq 0 \) and (5.10) on the right by \( x_* \geq 0 \), we get
\[
w_*R(\tilde{A})x_* + w_*u = w_*R(\tilde{b}), \tag{5.11}
\]
\[
w_*R(\tilde{A})x_* - vx_* = R(\tilde{c})x_* \tag{5.12}
\]

Subtraction of (5.12) from (5.11) yields
\[
w_*u + vx_* = w_*R(\tilde{b}) - R(\tilde{c})x_* \tag{5.13}
\]

Now for optimal \( x_* \) and \( w_* \), from Theorem 5.3 we have
\[
w_*R(\tilde{b}) = R(\tilde{c})x_* \tag{5.14}
\]

So, from (5.13) we obtain
\[
w_*u + vx_* = 0 \tag{5.15}
\]

Finally, substituting (5.15) into (5.13), we have
\[
(w_*R(\tilde{A}) - R(\tilde{c}))x_* + w_*(R(\tilde{b}) - R(\tilde{A})x_*) = 0 \tag{5.16}
\]
which implies
\[
(w_*A - \tilde{c})x_* + w_*(\tilde{b} - \tilde{A}x_*) = 0 \tag{5.17}
\]

The converse of the theorem follows similarly. \( \square \)

6. Conclusions

We used a linear ranking function to define the dual of fuzzy number linear programming primal problems. Similar to general linear programming, we presented several duality results.

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References