Multi-period asset allocation by stochastic dynamic programming

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Abstract

This study makes use of stochastic dynamic programming to set up a multi-period asset allocation model and derives an analytic formula for the optimal proportions invested in short and long bonds. Then maximum likelihood method is employed to estimate the relevant parameters. Finally, we implement the model through backward recursion algorithm to find numerically the optimal allocation of funds between short and long bonds for an investor with power utility and an investment horizon of ten years. Our results show that an investor will hold a larger proportion of short bond if his/her investment horizon gets shorter and/or if he/she is more risk averse.

Keywords: Multi-period asset allocation; Stochastic dynamic programming; Bellman function; Power utility; Two-factor Vasicek model; Backward recursion algorithm

1. Introduction

One of the major steps in the investment management process for investors is making asset allocation decisions to satisfy their investment objectives. That is, they must decide how their funds should be allocated among different assets in which they may invest. Asset allocation or portfolio choice is usually based on the assumption that investors use the mean–variance criterion of Markowitz [13] to allocate their funds among various assets. Such allocation is basically myopic or short-sighted in that it only optimizes over one period but ignores everything happening after the end of the one-period horizon. For asset allocation problems with more than one periods, Mossin [17] showed that myopic approach to investment is optimal only if investor has a logarithmic utility function. That is, in multi-period asset allocation problems, myopic strategies are suboptimal for other utility functions.

In multi-period asset allocation problems, an investor’s investment horizon is divided into $n$ periods, at the end of each of which return on the portfolio held over the period materializes and he/she can make a new decision on the composition of the portfolio to be held over the next period. His/her investment decisions are made with the objective of maximizing his/her expected utility of wealth at the end of the investment horizon.
horizon. Hence, an optimal asset allocation, besides diversified across assets, should also be diversified through time.

The extension of asset allocation problems from one period to multi-periods can be accomplished by dynamic programming. Pioneered by American Mathematician Bellman [2], dynamic programming is a technique for solving multi-period optimization problems by breaking them into one-period optimization problems. Hence, whereas asset allocation based on Markowitz criterion optimizes over one period, asset allocation based on dynamic programming optimizes over multi-periods.

Dynamic programming with random elements incorporated is known as stochastic dynamic programming [16]. Merton [14,15] was the first to apply this technology to consumption–portfolio allocation and developed a continuous-time model in which an investor optimizes his/her lifetime expected utility by selecting optimal consumption and portfolio choice. Specifically, for an investor with power utility, Merton derived explicit solutions for optimal consumption and optimal allocation between a risky asset (whose price follows the geometric Brownian motion) and a riskless asset (whose return is assumed constant).

The major drawback of Merton’s consumption–portfolio model is the assumption that the riskless asset has constant return. In other words, interest rate is assumed constant. In fact, interest rate fluctuates constantly and thus this assumption is clearly inconsistent with reality. In addition, many empirical studies, such as Schaefer [20], Stambaugh [21], and Litterman and Scheinkman [12], have found that at least two factors are necessary to explain the behavior of interest rates.

The aim of this study is threefold. First, as stated at the beginning, an optimal allocation, besides diversified across assets, should also be diversified through time. Hence, we study asset allocation in a multi-period context using the technique of stochastic dynamic programming. Second, for our formulation to accommodate the facts that interest rates are stochastic and that there are at least two factors in order to explain adequately their behavior, we employ a two-factor Vasicek [23] model to describe the evolution of the interest rates. Third, as bond securities have become increasingly popular among investors in the past 30 years [22], we investigate asset allocation between short and long bonds. Accordingly, this study uses stochastic dynamic programming to determine the optimal multi-period allocation between short and long bonds for an investor with an investment horizon of 10 years.

The rest of the paper proceeds as follows. In Section 2, stochastic dynamic programming is used to set up a multi-period asset allocation model and, eventually, an analytic formula is derived for the optimal proportion of wealth invested in short and long bonds. In Section 3, we employ maximum likelihood method to estimate the relevant parameters of the model. Section 4 shows how the model is implemented through backward recursion algorithm. In Section 5, we present the results for the optimal proportions invested in short and long bonds. Section 6 concludes this study.

2. Derivation of optimal allocation by stochastic dynamic programming

In this section, we make use of stochastic dynamic programming to set up our multi-period asset allocation model. We assume an investor allocates his/her wealth between short and long bonds so as to maximize his/her expected utility at the end of the investment horizon [0, T]. Assuming zero consumption before terminal time T and letting \( U[C(t), t] \) be a concave utility of consumption, the Bellman function \( I[W(t), t] \) is

\[
I[W(t), t] = \max E\left\{ \int_t^T U[C(s), s]ds \right\}.
\]

(1)

In Eq. (1), utility function plays an important role in constructing the Bellman function [10,16]. To have an explicit solution for their models, Samuelson [19], Merton [14,15], Richard [18], Brennan et al. [3], and Barberis [1] all assumed that the investor possesses power utility \( U[W] = \frac{W^{\gamma}}{\gamma} \) (where \( \gamma \) is the risk aversion parameter). Another advantage of using power utility is that it leads to an optimal solution which is independent of wealth. The use of power utility is supported by a generally cited paper of Friend and Blume [5], whose empirical results show that the typical utility function of an investor is characterized by decreasing absolute risk aversion and constant relative risk aversion. These properties are consistent with power utility. Hence, we use power utility in this study.
A two-factor Vasicek [23] interest rate model is used to describe the dynamics of the interest rates – one factor for the short rate and the other for the long rate. The Vasicek model has been extensively used by institutional investors as well as researchers [4,6,9,11] in valuing bond options, futures, futures options, and other types of contingent claims. Let \( r(t) \) be the short rate and \( l(t) \) be the long rate. Then the short rate and long rate are modeled as follows:

\[
\begin{align*}
\text{dr}(t) &= \alpha'(\beta' - r(t)) \text{d}t + \sigma' \text{d}Z'(t), \\
\text{dl}(t) &= \alpha'(\beta' - l(t)) \text{d}t + \sigma' \text{d}Z'(t),
\end{align*}
\]

where \( \text{d}Z'(t) \) and \( \text{d}Z'(t) \) are standard Wiener processes; \( \alpha' \) and \( \alpha' \) measure the strength of reversion to their respective mean levels, \( \beta' \) and \( \beta' \); \( \sigma' \) and \( \sigma' \) are the instantaneous volatilities of changes in the short rate and long rate. We assume that \( \text{d}Z'(t)\text{d}Z'(t) = \rho \text{d}t \), where \( \rho \) is the correlation between the random change in the short rate and that in the long rate.

Let \( P'(t) \) be the price of short bond and \( P'(t) \) be the price of long bond. We use short bond as the numeraire and assume that the short rate is the instantaneously riskless return of short bond. Since price and return for long bond respond more dramatically to changes in interest rates than those for short bond, long bond is considered more risky than short bond and thus it carries a risk premium. Accordingly, we assume that the expected return of long bond is the short rate plus a risk premium. Hence, the dynamics of their prices are

\[
\begin{align*}
\frac{\text{d}P'(t)}{P'(t)} &= r(t) \text{d}t, \\
\frac{\text{d}P'(t)}{P'(t)} &= [r(t) + \nu' \lambda'] \text{d}t + \nu' \text{d}Z'(t),
\end{align*}
\]

where \( \nu' \) is the volatility of \( P'(t) \) and \( \lambda' \) is the market price of interest rate risk. The above characterization of the price of long bond is consistent with the liquidity premium theory of the term structure. This theory predicts a rising yield curve. In fact, empirical evidence from US data in the past 40 years shows that the yield curve was upward-sloping most of the time and downward-sloping only in the early 1980s.

If we invest a fraction \( w(t) \) of wealth in the short bond and the rest \( 1 - w(t) \) in the long bond, the dynamics of wealth \( W(t) \) is

\[
\text{d}W(t) = W(t)w(t) \frac{\text{d}P'(t)}{P'(t)} + W(t)[1 - w(t)] \frac{\text{d}P'(t)}{P'(t)}. \tag{6}
\]

Substituting \( \frac{\text{d}P'(t)}{P'(t)} \) and \( \frac{\text{d}P'(t)}{P'(t)} \) of Eqs. (4) and (5) into Eq. (6) and simplifying, we have

\[
\text{d}W(t) = \{W(t)r(t) + W(t)[1 - w(t)]\nu' \lambda'] \text{d}t + W(t)[1 - w(t)]\nu' \text{d}Z'(t). \tag{7}
\]

For our case where we want to allocate an investor’s wealth between short and long bonds, we can rewrite \( I[W(t_t), t] = I[W(t, r, l, t)] \) in Eq. (1) as follows:

\[
I[W(t_t), r, l, t] = \max_w E_t \left\{ \int_{t}^{t+\text{d}t} U[C(s), s]\text{d}s + I[W + \text{d}W, r + \text{d}r, l + \text{d}l, t + \text{d}t] \right\}. \tag{8}
\]

Since no consumption takes place and no money is added to or withdrawn from the bond portfolio between time 0 and time \( T \), Eq. (8) can be simplified to

\[
I[W(t_t), r, l, t] = \max_w E_t \{I[W + \text{d}W, r + \text{d}r, l + \text{d}l, t + \text{d}t] \}. \tag{9}
\]

Expanding \( I[W + \text{d}W, r + \text{d}r, l + \text{d}l, t + \text{d}t] \) by Taylor’s theorem and suppressing the four arguments (i.e., \( W, r, l, t \)) of \( I[W, r, l, t] \) for simplicity to obtain

\[
I[W + \text{d}W, r + \text{d}r, l + \text{d}l, t + \text{d}t] = I + I_d \text{d}t + I_w \text{d}W + I_r \text{d}r + I_l \text{d}l + \frac{1}{2} I_{ww} \langle \text{d}W, \text{d}W \rangle + \frac{1}{2} I_{rr} \langle \text{d}r, \text{d}r \rangle + \frac{1}{2} I_l \langle \text{d}l, \text{d}l \rangle + I_{wr} \langle \text{d}W, \text{d}r \rangle + I_{wl} \langle \text{d}W, \text{d}l \rangle + I_{rl} \langle \text{d}r, \text{d}l \rangle. \tag{10}
\]
Given the expressions for dr, dl, and dW in Eqs. (2), (3), and (7), we have \( \langle dW, dW \rangle = W^2(1 - w(t))^2(v')^2 dt, \)
\( \langle dr, dr \rangle = (\sigma')^2 dt, \)
\( \langle dl, dl \rangle = (\sigma')^2 dt, \)
\( \langle dW, dr \rangle = W(1 - w(t)) \rho \sigma' dt, \)
\( \langle dW, dl \rangle = W(1 - w(t)) \rho \sigma' dt, \)
\( \langle dr, dl \rangle = \rho \sigma' dt. \)

Substituting dr, dl, dW, \( \langle dW, dW \rangle, \) \( \langle dr, dr \rangle, \) \( \langle dl, dl \rangle, \) \( \langle dW, dr \rangle, \) \( \langle dW, dl \rangle, \) and \( \langle dr, dl \rangle \) into Eq. (10) and taking expectation to get

\[
E_t \{ [W + dW, r + dr, l + dl, t + dt] \} = I + \left\{ I_t + I_W \{ W(t) + W(1 - w(t))v' \lambda'_t \} + I_r \{ \lambda' \{ \beta' - r(t) \} \} + I_l [\lambda' \{ \beta' - l(t) \}] + \frac{1}{2} I_{WW} W^2(1 - w(t))^2(v')^2 + \frac{1}{2} I_{rr} (\sigma')^2 + \frac{1}{2} I_{ll} (\sigma')^2 + I_{Wl} W(1 - w(t)) \rho \sigma' + I_{Wl} W(1 - w(t))v' \lambda' + I_{rl} \rho \sigma' \lambda' \right\} dt.
\]

(11)

Substituting Eq. (11) into Eq. (9) and then simplifying, the following Bellman optimality equation [10,16] is obtained

\[
0 = \max_w \left\{ I_t + I_W \{ r(t) + (1 - w(t))v' \lambda'_t \} + I_r \{ \lambda' \{ \beta' - r(t) \} \} + I_l [\lambda' \{ \beta' - l(t) \}] + \frac{1}{2} I_{WW} W^2(1 - w(t))^2(v')^2 + \frac{1}{2} I_{rr} (\sigma')^2 + \frac{1}{2} I_{ll} (\sigma')^2 + I_{Wl} W(1 - w(t)) \rho \sigma' + I_{Wl} W(1 - w(t))v' \lambda' + I_{rl} \rho \sigma' \lambda' \right\}.
\]

(12)

To solve the Bellman equation, let \( I[W, r, l, t] = \frac{W^2}{C} K[r, l, t], \) where \( K[r, l, t] \) is any positive function. With no consumption until time \( T, \) we have at time \( T: \) \( I[W, r, l, T] = \max_{E_T} [U[W(T)]] = U[W(T)] = \frac{W^2}{C}. \) Hence, \( K[r, l, T] = 1. \) Suppressing the three arguments (i.e., \( r, l, \) and \( t \)) of \( K[r, l, t] \) for simplicity and replacing \( I-related terms \) by \( K-related terms \) in Eq. (12) to obtain

\[
0 = \max_w \left\{ \frac{1}{\gamma} K_t + K \{ r(t) + (1 - w(t))v' \lambda'_t \} + \frac{1}{\gamma} K_r \{ \lambda' \{ \beta' - r(t) \} \} + \frac{1}{\gamma} K_l [\lambda' \{ \beta' - l(t) \}] + \frac{1}{2} \gamma K_{rr} (\sigma')^2 + \frac{1}{2} \gamma K_{ll} (\sigma')^2 + \frac{1}{2} \gamma K_{Wl} \rho \sigma' \lambda' + K_r (1 - w(t)) \rho \sigma' + K_l (1 - w(t)) v' \lambda' + \frac{1}{\gamma} K_{rl} \rho \sigma' \lambda' \right\}.
\]

(13)

The term \( W^2 \) is eliminated from the Bellman optimality equation because power utility is used. In other words, optimal proportion \( w(t) \) is independent of wealth \( W(t). \) Simplifying Eq. (13) gives

\[
\max_w \{ a(1 - w(t))^2 + b(1 - w(t)) + c \} = 0,
\]

(14)

where \( a = \frac{1}{2} (\gamma - 1) K(v')^2, b = K \lambda' l + H \rho \sigma' \lambda' + H v' \lambda', c = \frac{1}{\gamma} K_t + K_r(t) + \frac{1}{\gamma} \{ K_r \{ \lambda' \{ \beta' - r(t) \} \} + K_l [\lambda' \{ \beta' - l(t) \}] \} + \frac{1}{\gamma} \{ K_{rr} (\sigma')^2 + K_{ll} (\sigma')^2 \} + \frac{1}{\gamma} \{ K_{Wl} \rho \sigma' \lambda' \} + \frac{1}{\gamma} \{ K_{rl} \rho \sigma' \lambda' \}. \)

By the first-order condition for a maximum in Eq. (14), the following optimal proportion \( w^*(t) = w^*[r, l, t] \) invested in short bond is obtained

\[
w^*[r, l, t] = 1 + \frac{b}{2a} = 1 + \left[ \frac{K \lambda' l + K_r \rho \sigma' + K_l \lambda'}{(\gamma - 1)Kv'} \right].
\]

(15)

Correspondingly, the optimal proportion invested in long bond is simply \( 1 - w^*(t). \)

3. Parameter estimation by maximum likelihood method

To find numerically the optimal proportions in short and long bonds based on Eq. (15), we have to estimate first the relevant parameters in Eqs. (2), (3), and (5) of Section 2. In this study, daily data, retrieved from online US Federal Reserve Economic Data (FRED) series, used for our estimation are three-month US
Treasury bill rate and 20-year US Treasury constant maturity rate from 2 January 1962 to 29 December 2006—a total of 11,289 observations for each of the two data sets. We use three-month rate to represent the short rate \( r(t) \) and 20-year rate to represent the long rate \( l(t) \). Given the fact that a coupon of $4 is paid every 6 months, the price (i.e., \( P(t) \)) of the long bond is computed as follows:

\[
P'(t) = \frac{\$4}{\left(1 + \frac{r(t)}{2}\right)^2} + \frac{\$4}{\left(1 + \frac{l(t)}{2}\right)^2} + \cdots + \frac{\$4}{\left(1 + \frac{l(t)}{2}\right)^{40}} + \$1,000 \tag{16}
\]

The estimation is done by maximum likelihood (ML) method [7]. ML estimation is based on large sample asymptotic estimators, which are asymptotically normally distributed. Since we have a total of 11,289 observations for each of the three data sets, the ML estimates obtained should be accurate even if the distribution from which the data were taken is not normal.

The diffusion processes in Eqs. (2) and (3) can be expressed in discrete form as follows:

\[
r(t + \Delta t) - r(t) = x' [\beta' - r(t)] \Delta t + \sigma' \varepsilon' \sqrt{\Delta t},
\]

\[
l(t + \Delta t) - l(t) = x' [\beta' - l(t)] \Delta t + \sigma' \varepsilon' \sqrt{\Delta t},
\]

where \( \varepsilon' \) and \( \varepsilon' \) are independent standard normal deviates. Hence, we have that \( r(t + \Delta t) - r(t) - x' [\beta' - r(t)] \Delta t = \sigma' \varepsilon'(t) \sqrt{\Delta t} \) is distributed \( N[0, (\sigma')^2 \Delta t] \) and \( l(t + \Delta t) - l(t) - x' [\beta' - l(t)] \Delta t = \sigma' \varepsilon'(t) \sqrt{\Delta t} \) is distributed \( N[0, (\sigma')^2 \Delta t] \). With \( n \) independent observations, the logarithms of the likelihood function \( L[x', \beta', \sigma'] \) (as a function of \( x', \beta' \), and \( \sigma' \)) and of the likelihood function \( L[x', \beta', \sigma'] \) (as a function of \( x', \beta' \), and \( \sigma' \)) can be written as

\[
L(x', \beta', \sigma') = -\frac{n}{2} \ln(\sigma')^2 \Delta t - \frac{n}{2} \ln(2\pi) - \sum_{i=1}^{n} \left\{ r(t + \Delta t) - r(t) - x' [\beta' - r(t)] \Delta t \right\}^2 / 2(\sigma')^2 \Delta t,
\]

\[
L(x', \beta', \sigma') = -\frac{n}{2} \ln(\sigma')^2 \Delta t - \frac{n}{2} \ln(2\pi) - \sum_{i=1}^{n} \left\{ l(t + \Delta t) - l(t) - x' [\beta' - l(t)] \Delta t \right\}^2 / 2(\sigma')^2 \Delta t.
\]

The diffusion process in Eq. (5) can be expressed in discrete form as

\[
\frac{\Delta P'(t)}{P'(t)} = [r(t) + \psi' \lambda'] \Delta t + \psi' \varepsilon' \sqrt{\Delta t},
\]

where \( \varepsilon' \) is a standard normal deviate. Given that \( E[\Delta P(t)/P(t)] = r(t) + \psi' \lambda' \) and \( \text{var}[\Delta P(t)/P(t)] = (\psi')^2 \), the logarithm of price relatives \( x(t) = \ln[P(t)/P(t-1)] \) is normally distributed with mean \( = r(t) + \psi' \lambda' - \psi' \lambda' \Delta t \) and variance \( = (\psi')^2 \). With \( n \) independent observations, the logarithm of the likelihood function \( L[\psi', \lambda'] \), viewed as a function of \( \psi' \) and \( \lambda' \), can be written as

\[
L(\psi', \lambda') = -\frac{n}{2} \ln(\psi')^2 \Delta t - \frac{n}{2} \ln(2\pi) - \sum_{i=1}^{n} \left\{ x(t) - [r(t) + \psi' \lambda' - \psi' \lambda' \Delta t] \right\}^2 / 2(\psi')^2 \Delta t.
\]

ML estimates are found by maximizing Eq. (19) with respect to \( x', \beta', \) and \( \sigma' \); Eq. (20) with respect to \( x', \beta', \) and \( \sigma' \); and Eq. (22) with respect to \( \psi' \) and \( \lambda' \). Table 1 presents the ML estimates for the eight parameters.

**Table 1**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
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<td>0.2633</td>
</tr>
<tr>
<td>( x' )</td>
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<td>0.0188</td>
</tr>
<tr>
<td>( \sigma' )</td>
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<td>0.0212</td>
</tr>
<tr>
<td>( \sigma' )</td>
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<td>0.0187</td>
</tr>
<tr>
<td>( \psi' )</td>
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<td>0.0873</td>
</tr>
<tr>
<td>( \lambda' )</td>
<td>0.2129</td>
<td>0.1142</td>
</tr>
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</table>
4. Numerical implementation by backward recursion algorithm

In this section, we determine through backward recursion algorithm the optimal proportions in short and long bonds based on Eq. (15). Accordingly, we divide the investment horizon [0, T] into m equal periods of length $\Delta t = \frac{T}{m}$, where $T$ is set to be 10 years. We begin with the last period and work backward in time toward the first period. At the start of each period or time $t$ (where $t = 0, \Delta t, 2\Delta t, \ldots, m\Delta t$), a two-dimensional grid with short rate $r(t)$ and long rate $l(t)$ as the two axes is constructed, where $r(t)$ and $l(t)$ range from 0.00 to 0.20. Point $(i, j)$ on the grid at time $t$ corresponds to $r(t) = i \Delta r$ and $l(t) = j \Delta l$, where $i = j = 0, 1, 2, \ldots, 20$ and $\Delta r = \Delta l = 0.01$. Our objective is to find numerically the optimal proportion $w^*(t)$ for each point on the grid at each time $t$.

To find the optimal proportion $w^*(t)$, we have to calculate first the values of $K[r, l, t]$. Substituting $w^*[r, l, t] = 1 + \frac{b}{\Delta t}$ into Eq. (14) to get

$$-K_t = -\left\{ \frac{K[r, l, t] - K[r, l, t - \Delta t]}{\Delta t} \right\} = \left[ c - \frac{1}{\gamma} K_l \right] \gamma - \frac{b^2 \gamma}{4a}.$$

Simplifying Eq. (23), we have

$$K[r, l, t - \Delta t] = K[r, l, t] + \Delta t \left\{ \left[ c - \frac{1}{\gamma} K_l \right] \gamma - \frac{b^2 \gamma}{4a} \right\}. \quad (24)$$

Substituting $a$, $b$, and $c$ of Eq. (14) into Eq. (24), we obtain

$$K[r, l, t - \Delta t] = K[r, l, t] + \Delta t \left\{ \gamma K_r(t) + K_r[z' \beta^r(t)] + K_l[z' \beta^l(t)] \right\} + \gamma (\rho \sigma^r \sigma^l \Delta t + \frac{1}{2} \Delta t \left\{ K_{rr}(\sigma^r)^2 + K_{ll}(\sigma^l)^2 - \frac{\gamma (K_r^2 \sigma^r \sigma^l + K_r \sigma^r \sigma^l + K_l \sigma^l \sigma^r)}{(\gamma - 1)K} \right\}. \quad (25)$$

We use explicit finite difference method to find the partial derivatives of $K[r, l, t]$. The partial derivatives are approximated by the following discrete operators [8]:

$$K_r = K[r, l, t] \approx \frac{1}{\Delta r} \left\{ K[r + \Delta r, l, t] - K[r - \Delta r, l, t] \right\}, \quad (26)$$

$$K_l = K[r, l, t] \approx \frac{1}{\Delta l} \left\{ K[r, l + \Delta l, t] - K[r, l - \Delta l, t] \right\}, \quad (27)$$

$$K_{rr} = K[r, l, t] \approx \frac{1}{\Delta r^2} \left\{ K[r + \Delta r, l, t] - 2 K[r, l, t] + K[r - \Delta r, l, t] \right\}, \quad (28)$$

$$K_{ll} = K[r, l, t] \approx \frac{1}{\Delta l^2} \left\{ K[r, l + \Delta l, t] - 2 K[r, l, t] + K[r, l - \Delta l, t] \right\}. \quad (29)$$

<table>
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<th>Year</th>
<th>$\gamma = -0.5$</th>
<th>$\gamma = -2.0$</th>
<th>$\gamma = -5.0$</th>
<th>$\gamma = -10.0$</th>
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<td>0.6243–0.6559</td>
<td>0.7082–0.7700</td>
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<tr>
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<td>0.6260–0.6567</td>
<td>0.7099–0.7704</td>
</tr>
<tr>
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<td>0.6286–0.6580</td>
<td>0.7127–0.7711</td>
</tr>
<tr>
<td>3</td>
<td>0.0197–0.0210</td>
<td>0.4310–0.4404</td>
<td>0.6324–0.6600</td>
<td>0.7169–0.7723</td>
</tr>
<tr>
<td>4</td>
<td>0.0217–0.0228</td>
<td>0.4351–0.4436</td>
<td>0.6379–0.6630</td>
<td>0.7233–0.7742</td>
</tr>
<tr>
<td>5</td>
<td>0.0246–0.0256</td>
<td>0.4410–0.4484</td>
<td>0.6457–0.6676</td>
<td>0.7326–0.7774</td>
</tr>
<tr>
<td>6</td>
<td>0.0287–0.0295</td>
<td>0.4494–0.4553</td>
<td>0.6565–0.6745</td>
<td>0.7459–0.7826</td>
</tr>
<tr>
<td>7</td>
<td>0.0345–0.0350</td>
<td>0.4613–0.4655</td>
<td>0.6725–0.6852</td>
<td>0.7647–0.7913</td>
</tr>
<tr>
<td>8</td>
<td>0.0427–0.0430</td>
<td>0.4782–0.4805</td>
<td>0.6946–0.7018</td>
<td>0.7907–0.8060</td>
</tr>
<tr>
<td>9</td>
<td>0.0544–0.0545</td>
<td>0.5020–0.5027</td>
<td>0.7253–0.7276</td>
<td>0.8261–0.8309</td>
</tr>
</tbody>
</table>

Table 2
Ranges of optimal proportions invested in short bond
5. Numerical results

In this section, we present the optimal proportions in short bond given that the risk aversion parameter $\gamma$ is $-0.5$, $-2.0$, $-5.0$, and $-10.0$. Note that an investor with a larger $-\gamma$ means that he/she is more risk averse. Table 2 shows the ranges of optimal proportions at different times and for different values of $\gamma$. Two obvious phenomena can be observed from Table 2. The first is that, for each risk aversion parameter, the optimal proportions increase as the investment horizon gets shorter. That is, an investor would allocate a larger proportion of his/her funds to short bond if he/she is more risk averse.

The second phenomenon is that the risk aversion parameter $\gamma$ has a notable impact on the optimal proportions in short bond. Specifically, other things held constant, the optimal proportions become larger and larger as $-\gamma$ gets larger in magnitude. That is, an investor would invest more in short bond if he/she is more risk averse. For example, at year 5, the optimal proportions range from 0.6243 to 0.6559 at year 0, from 0.6457 to 0.6676 at year 5, and from 0.7253 to 0.7276 at year 9. The reason for such allocation of funds is that an investor with a distant investment horizon would place a larger proportion of his/her funds in long bond to hedge against adverse changes in the future investment opportunity set; however, as his/her horizon is approached, he/she would place a larger proportion of his/her funds in short bond because now variability in the value of the investment opportunity set has diminished.

The algorithm of backward recursion begins at time $T-1 = (m-1)\Delta t$. For each point on the grid at time $T-1$, we calculate the values of $K[r,l,T-1]$ from the values of $K[r,l,T]$, where $K[r,l,T] = 1$ at time $T$ for all values of short and long rates. The values of $w^*[r,l,T-1]$ are then calculated from the values of $K[r,l,T-1]$ using Eq. (15). After this is done for time $T-1$, the values of $K[r,l,T-2]$ and $w^*[r,l,T-2]$ are calculated for each point on the grid at time $T-2$ from the values of $K[r,l,T-1]$. This procedure is done repeatedly until time 0 is reached. To ensure satisfactory convergence, a small value of 0.001 is used for $\Delta t$. Hence, for $T = 10$ years, the number of time steps is $\frac{T}{\Delta t} = 10,000$.

6. Conclusion

In this paper, we make use of stochastic dynamic programming to formulate a multi-period asset allocation model and derive an analytic formula for the optimal proportions in short and long bonds. Then we employ maximum likelihood method to estimate the relevant parameters of the model. Finally, we implement the model through backward recursion algorithm to find numerically the optimal allocation of funds between short and long bonds for an investor with power utility and an investment horizon of ten years. Our results show that an investor will allocate a larger proportion of his/her funds to short bond if his/her investment horizon gets shorter and/or if he/she is more risk averse.

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References